Section 7, 10/14/19

Fisher Information

Given some density $f(X|\theta)$, recall that we define the score function of the likelihood as

$$\frac{\partial}{\partial\theta} log\, f(X|\theta)$$

And the **Fisher Information** as the second moment of the score function

$$I(\theta) = E\big[\big(\frac{\partial}{\partial\theta} log\, f(X|\theta)\big)^2\big]$$

Lemma A (page 276)

Given appropriate smoothness conditions of the density, we can rewrite

$$I(\theta) = -E\big[\frac{\partial^2}{\partial\theta^2} log\, f(X|\theta)\big]$$

The proof for this is as follows. Consider the first moment of the score function

$$E\big[\frac{\partial}{\partial\theta} log\, f(X|\theta)\big] = \int \Big[\frac{\partial}{\partial\theta} log\, f(x|\theta)\Big] f(x|\theta) dx$$

We use two properties. The first is that under appropriate smoothness conditions, the integral and the derivative are interchangeable. The second property is the identity:

$$\Big[\frac{\partial}{\partial\theta} log\, f(x|\theta)\Big] f(x|\theta) = \frac{\partial}{\partial\theta} f(x|\theta)$$

Thus,

$$E\big[\frac{\partial}{\partial\theta} log\, f(X|\theta)\big] = \int \Big[\frac{\partial}{\partial\theta} log\, f(x|\theta)\Big] f(x|\theta) dx = \int \frac{\partial}{\partial\theta} f(x|\theta) dx = \frac{\partial}{\partial\theta} \int f(x|\theta) dx = \frac{\partial}{\partial\theta}(1) = 0$$

So, we can take the partial derivative of each side

$$\begin{aligned} 0 &= \frac{\partial}{\partial\theta} \int \Big[\frac{\partial}{\partial\theta} log\, f(x|\theta)\Big] f(x|\theta) dx \\ &= \int \frac{\partial}{\partial\theta} \bigg(\Big[\frac{\partial}{\partial\theta} log\, f(x|\theta)\Big] f(x|\theta)\bigg) dx \\ &= \int \Big[\frac{\partial^2}{\partial\theta^2} log\, f(x|\theta)\Big] f(x|\theta) dx + \int \Big[\frac{\partial}{\partial\theta} log\, f(x|\theta)\Big]^2 f(x|\theta) dx \end{aligned}$$

The latter expression is the second moment of the score function, the Fisher Information! Rearranging,

$$-\int \left[\frac{\partial^2}{\partial\theta^2} log\, f(x|\theta)\right] f(x|\theta)dx = \int \left[\frac{\partial}{\partial\theta} log\, f(x|\theta)\right]^2 f(x|\theta)dx$$

$$-E\big[\frac{\partial^2}{\partial\theta^2} log\, f(X|\theta)\big] = I(\theta)$$