Section 1, 9/2/19

Multivariate Change of Basis

Let $(X, Y) \sim F_{XY}$ and X, Y are mapped onto U, V by some invertible transformation such that

$$u = g_1(x, y) \quad v = g_2(x, y)$$
$$x = h_1(u, v) \quad y = h_2(u, v)$$

Then (assuming that g_1, g_2 have continuous partial derivatives) and that the Jacobian

$$J(x, y) = \det \begin{bmatrix} \frac{\partial g_1}{\partial x} & \frac{\partial g_1}{\partial y} \\ \frac{\partial g_2}{\partial x} & \frac{\partial g_2}{\partial y} \end{bmatrix} = \left(\frac{\partial g_1}{\partial x}\right)\left(\frac{\partial g_2}{\partial y}\right) - \left(\frac{\partial g_2}{\partial x}\right)\left(\frac{\partial g_1}{\partial y}\right) \neq 0$$

PROPOSITION **A**

Under the assumptions just stated, the joint density of U and V is

$$f_{UV}(u, v) = f_{XY}(h_1(u, v), h_2(u, v))|J^{-1}(h_1(u, v), h_2(u, v))|$$

for (u, v) such that $u = g_1(x, y)$ and $v = g_2(x, y)$ for some (x, y) and 0 elsewhere. ■

In proposition A, the inverse Jacobian term compensates for how areas/volumes/higher dimensional regions change under the transformation.

Ex.

Suppose that X_1 and X_2 are independent standard normal random variables and that

$$Y_1 = X_1$$
$$Y_2 = X_1 + X_2$$

We will show that the joint distribution of Y_1 and Y_2 is bivariate normal. The Jacobian of the transformation is simply

$$J(x, y) = \det \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = 1$$

Since the inverse transformation is $x_1 = y_1$ and $x_2 = y_2 - y_1$, from Proposition A the joint density of Y_1 and Y_2 is

$$f_{Y_1 Y_2}(y_1, y_2) = \frac{1}{2\pi} \exp\left[-\frac{1}{2}[y_1^2 + (y_2 - y_1)^2]\right]$$
$$= \frac{1}{2\pi} \exp\left[-\frac{1}{2}(2y_1^2 + y_2^2 - 2y_1 y_2)\right]$$

f_{Y_1,Y_2} can be seen as a bivariate normal distribution

Bivariate Normal Distribution

Normal distribution

$f_X(x) = \frac{1}{\sigma\sqrt{2\pi}} exp(-\frac{(x-\mu)^2}{2\sigma^2})$

Bivariate normal

For means μ_x, μ_y, variances σ_x, σ_y and population correlation coefficient $\rho = \frac{cov(X,Y)}{\sigma_x \sigma_y}$

As a reminder, $cov(X,Y) = E[(X - E[X])(Y - E[Y])] = E[XY] - E[X]E[Y]$

$$f(x, y) = \frac{1}{2\pi\sigma_X\sigma_Y\sqrt{1-\rho^2}} \exp\left(-\frac{1}{2(1-\rho^2)}\left[\frac{(x-\mu_X)^2}{\sigma_X^2} + \frac{(y-\mu_Y)^2}{\sigma_Y^2} - \frac{2\rho(x-\mu_X)(y-\mu_Y)}{\sigma_X\sigma_Y}\right]\right)$$

Another way to write the bivariate normal is in quadratic form

Let

$$\mathrm{x} = [x, y]^T$$
$$\mu = [\mu_x, \mu_y]^T$$
$$\Sigma = \begin{pmatrix} \sigma_x^2 & \sigma_{xy} \\ \sigma_{xy} & \sigma_y^2 \end{pmatrix}$$

Instead of σ_{xy} you may see $\rho\sigma_x\sigma_y$ sometimes as they are equivalent. Then

$f_{XY}(x, y) = \frac{1}{\sqrt{(2\pi)^k|\Sigma|}} exp[-\frac{1}{2}(\mathrm{x} - \mu)^T\Sigma^{-1}(\mathrm{x} - \mu)]$ This form generalizes to multivariate gaussians of higher dimension.

Sometimes we are interested in the conditional density, however.

Recall the conditional density $f_{Y|X}(y|x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$

Applying this to the bivariate normal distribution and reducing, we get:

$$f_{Y|X}(y|x) = \frac{1}{\sigma_Y\sqrt{2\pi(1-\rho^2)}} \exp\left(-\frac{1}{2}\frac{\left[y - \mu_Y - \rho\frac{\sigma_Y}{\sigma_X}(x - \mu_X)\right]^2}{\sigma_Y^2(1-\rho^2)}\right)$$

Notice that this is exactly a univariate normal distribution whose mean and variance are dependent on ρ and μ_x. This is easy to visualize on the plot of a multivariate normal. In the case that $\rho = 0$, i.e. they are independent, we recover f_Y as we would expect.